On the Cauchy problem of generalized complex Ginzburg-Landau equation in three dimensions

LI Donglong[1,2] and GUO Boling[3]

(1. Guangxi Institute of Technology, Liuzhou 545006, China; 2. Graduate School of China Academy of Engineering Physics, Beijing 100088, China; 3. Institute of Applied Physics and Computational Mathematics, P. O. Box 8009, Beijing 100088, China)

Received January 14, 2003; revised March 4, 2003

Abstract The generlized complex Ginzburg-Landau equation (CGL) under periodic boundary condition is studied. The existence of global solution for this equation is established under appropriate assumption on a nonlinear σ, which rigorously establishes the foundation for further investigation of this type of model.

Keywords: generalized complex Ginzburg-Landau equation, global solution.

The description of spatial pattern formation or chaotic dynamics in continuum systems, particularly in the fluid dynamical system, is a challenging task in theoretical physics and applied mathematics. Due to the complexity of the corresponding nonlinear evolution equations, simpler model equations for which the mathematical issues can be solved with greater success, have been derived. The complex Ginzberg-Landau equation is one of these equations. It models the evolution of the amplitude of perturbations to steady-state solutions at the onset of instability. It is a particularly interesting model because it is a dissipative version of the nonlinear Schrödinger equation—A Hamiltonian equation which can possess solutions that form localized singularities in finite time.

The complex Ginzburg-Landau equation (CGL) is of the form:

$$u_t = \rho u + (1 + i\nu) u_{xx} - (1 + i\mu) | u |^{2\sigma} u .$$

Doering et al.[1], Ghidaglia and Héorn[2] studied the finite dimensional global attractor and related dynamic issues for the one or two spatial dimensional GLE with cubic nonlinearity ($\sigma = 1$). Bartuccelli et al.[3] dealt with the "soft" and "hard" turbulent behavior for this equation. Doering[4] presented the existence and uniqueness of global weak and strong solutions for this equation in all spatial dimensions and for all degree of nonlinearity $\sigma > 0$.

The generalized complex Ginzburg-Landau equation (CGL) is of the form:

$$u_t = \rho u + (1 + i\nu)\Delta u - (1 + i\mu) | u |^{2\sigma} u + \alpha\lambda_1 \cdot \nabla(| u |^2 u) + \beta(\lambda_2 \cdot \nabla) | u |^2 .$$

Guo and Wang[5] studied finite dimensional behavior for this equation.

We consider the generalized complex Ginzburg-Landau equation in three-dimension space

$$\begin{aligned} u_t = \rho u - \Delta\varphi(u) - (1 + i\gamma)\Delta u - \nu\Delta^2 u \\ - (1 + i\mu) | u |^{2\sigma} u + \alpha\lambda_1 \cdot \nabla(| u |^2 u) \\ + \beta(\lambda_2 \cdot \nabla) | u |^2 . \end{aligned} \quad (1)$$

The equation will be supplemented with the following usual initial-value and boundary conditions: the initial-value condition

$$u(x,0) = u_0(x) \text{ in } \Omega; \quad (2)$$

the periodic boundary condition

$$\Omega = (0, L_1) \times (0, L_2) \times (0, L_3), \quad u \text{ is } \Omega\text{-periodic}. \quad (3)$$

The main work of this paper is to establish the existence and uniqueness of global solution for Eqs. (1) ~ (3) under appropriate assumpation on a nonlinear σ.

Our assumptions on σ, γ, μ, ν are (**A**):

(i) σ, γ, μ satisfy

$$3 \leqslant \sigma < \min\left\{ \frac{1}{\sqrt{1 + \mu^2} - 1}, \frac{1}{\sqrt{1 + \gamma^2} - 1} \right\}.$$

(ii) There are positive numbers $\delta > 0$ and choosing suitable ν such that

$$\frac{\delta^2}{2} - \frac{1}{\nu}((1 + \nu\delta^2)^2 + \mu^2) \geqslant 0.$$

E-mail: lidl@21cn.com, gbl@mail.math.iapcm.cn

In the following, we denote norm of $H^k_{per}(\Omega)$ as same as the norm of usual Soblev space with the norm $\| u \|_{H^k_{per}(\Omega)} = \left(\sum_{|s|<m} \| D^s u \|^2 \right)^{\frac{1}{2}}$ and by $\| \cdot \|$ the norm of $H = L^2_{per}(\Omega)$ with usual inner product $(\cdot, \cdot)$, $\| \cdot \|_p$ denotes the norm of $L^p_{per}(\Omega)$ for $1 \leqslant p \leqslant \infty$ ($\| \cdot \|_2 = \| \cdot \|$).

1 Existence and uniqueness of the solution

First, we discuss local existence of the initial-value problems (1)~(3).

The following inequality is needed:

Lemma 1. (Uniform Gronwall's inequality)

Let g, h, y be three positive locally integrable functions on $[t_0, \infty)$, such that y' is locally integrable on $[t_0, \infty)$, which satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}t} < gy + h, \quad \text{for } t \geqslant t_0,$$

$$\int_t^{t+r} g(s)\mathrm{d}s \leqslant a_1, \quad \int_t^{t+r} h(s)\mathrm{d}s \leqslant a_2,$$

$$\int_t^{t+r} y(s)\mathrm{d}s \leqslant a_3, \quad \text{for } t \geqslant t_0,$$

where r, a_1, a_2 and a_3 are positive constants. Then

$$y(t+r) \leqslant \left(\frac{a_3}{r} + a_2 \right) \exp(a_1), \quad \text{for } t \geqslant t_0.$$

Lemma 2. (Gagliardo-Nirenberg's inequality in a 3-dimension space)

$$\| \nabla^j u \|_p \leqslant c \| \nabla^m u \|_r^a \| u \|_q^{1-a},$$

where

$$\frac{1}{p} = \frac{j}{3} + a\left(\frac{1}{r} - \frac{m}{3} \right) + \frac{1-a}{q}$$

with $1 \leqslant q$, $r < \infty$. Two further restrictions are $0 \leqslant j < m$ and $\frac{j}{m} \leqslant a < 1$.

We define a linear operator in $X = L^2$, $Au = \nu \Delta^2 u$ with definition $D(A) = H^4 \cap H^2_0$.

By Lumer-phillips theorem the linear operator A is the infinitesimal generator of a continuous semigroup of contraction $S(t) = \exp At$ for $t > 0$ (see Refs. [6,7]). Let

$$\begin{aligned} N(u) = {} & \rho u - (1 + i\gamma)\Delta u + \Delta\varphi(u) \\ & - (1 + i\mu) | u |^{2\sigma} u + \alpha\lambda_1 \cdot \nabla(| u |^2 u) \\ & + \beta(\lambda_2 \cdot \nabla) | u |. \end{aligned}$$

Thus we rewrite (1) in a shortened form

$$u_t = Au + N(u).$$

In order to obtain the existence of local solution of the initial-value problems (1)~(3) for every $u_0 \in H^2_{per}(\Omega)$, we need the properties of the nonlinear operator N.

Lemma 3. The nonlinear mapping $N(u)$ maps $H^2_{per}(\Omega)$ into $L^2_{per}(\Omega)$ and satisfies for $u, v \in H^2_{per}(\Omega)$,

$$\| N(u) \| \leqslant C \| u \|_{H^2_{per}(\Omega)},$$

$$\begin{aligned} & \| N(u) - N(v) \| \\ & \leqslant C(\| u \|_{H^2_{per}(\Omega)}, \| v \|_{H^2_{per}(\Omega)}) \| u - v \|_{H^2_{per}(\Omega)}. \end{aligned}$$

By using G-N's inequality, proof of lemma is not difficult, we omit it here.

Therefore we have

Lemma 4. For every $u \in H^2_{per}(\Omega)$ there exists a unique solution u of the initial-value problems (1)~(3) on a finite time interval $t \in [0, T_{\max})$ so that

$$\begin{aligned} u \in {} & C^1([0, T_{\max}): L^2_{per}(\Omega)) \\ & \cap C([0, T_{\max}): L^2_{per}(\Omega)) \end{aligned}$$

with the property that

$$T_{\max} = \infty \text{ or if } T_{\max} < \infty$$
$$\text{then } \lim_{t \to T_{\max}} \| u(t) \|_{H^2_{per}(\Omega)} \to \infty.$$

Futhermore, by the results of [7], we know that

$$\begin{aligned} u \in {} & C^1([0, T_{\max}): H^2_{per}(\Omega)) \\ & \cap C([0, T_{\max}): H^2_{per}(\Omega)). \end{aligned}$$

Proof. The result follows from [6] (theorem 3.3.3, theorem 3.3.4, theorem 3.5.2) and [7] (theorem 6.3.1).

2 A priori estimate

In order to show the global existence for all $t > 0$, we need to establish some time-uniform a priori estimate on $u(t)$ in $L^2_{per}(\Omega)$ and $H^1_{per}(\Omega)$, $H^2_{per}(\Omega)$.

Lemma 5. Assume that $u_0 \in L^2_{per}(\Omega)$, and $\varphi'(u) \leqslant 0$. Then for the solution $u(t)$ of (1)~(3), we have

$$\| u \|^2 \leqslant K_1(\| u_0 \|, T), \quad t \in [0, T]; \tag{4}$$

$$\int_t^{t+1} \| \Delta u \|^2 \mathrm{d}t \leqslant K_1(\| u_0 \|, T), \quad t \in [0, T]; \tag{5}$$

$$\int_t^{t+1} \| u \|_{2\sigma+2}^{2\sigma+2} dt \leqslant K_1(\| u_0 \|, T),$$
$$t \in [0, T]; \qquad (6)$$

where K_1 depends on T, and T depends on the data $\sigma, \rho, \gamma, \mu, \nu, \alpha, \beta$ and R when $\| u_0 \| < R$.

Proof. Taking inner product in $L^2_{per}(\Omega)$ of (1) with u and then taking the real part of the resulting identity, it follows

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} \| u \|^2 = {} & \rho \| u \|^2 + \| \nabla u \|^2 - \nu \| \Delta u \|^2 \\ & - (\Delta\varphi(u), u) - \int | u |^{2\sigma+2} \\ & + Re(u, \alpha\lambda_1 \cdot \nabla(| u |^2 u) \\ & + \beta(\lambda_2 \cdot \nabla u) | u |^2). \end{aligned} \qquad (7)$$

Obviously, we have

$$Re(u, \alpha\lambda_1 \cdot \nabla(| u |^2 u) + \beta(\lambda_2 \cdot \nabla u) | u |^2) = 0.$$

For the fourth term of (7) on the right side, we have

$$\begin{aligned}(\Delta\varphi(u), u) &= -(\nabla\varphi(u), \nabla u) \\ &= -(\varphi'(u)\nabla u, \nabla u) \\ &= -(\varphi'(u), \| \nabla u \|^2) \geqslant 0,\end{aligned}$$

and note that

$$\begin{aligned}\| \nabla u \|^2 &\leqslant \| \Delta u \| \| u \| \\ &\leqslant \frac{\nu}{2} \| \Delta u \|^2 + \frac{1}{2\nu} \| u \|^2.\end{aligned}$$

Thus synthesizing the above inequality we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt} \| u \|^2 + \frac{\nu}{2} \| \Delta u \|^2 + \int | u |^{2\sigma+2} \\ &\leqslant \left(\rho + \frac{1}{2\nu}\right) \| u \|^2.\end{aligned} \qquad (8)$$

Therefore, applying Gronwall's inequality we obtain (4). After integrating (8) with t between t and $t+1$, we obtain (5), (6).

Here and after we denote c as any constant depending on the data $\sigma, \rho, \gamma, \mu, \nu, \alpha, \beta$.

Lemma 6. Under the assumptions of (**A**), and when

$$| \varphi'(u) | \leqslant C | u |^p, \quad 0 \leqslant p < \frac{4}{3},$$

for the solution of the problems (1)~(3) we have

$$\| \nabla u \|^2 \leqslant K_2(\| u_0 \|_{H^1_{per}}, T), \quad t \in [0, T],$$

where K_2 depends on data and T, and T depends on R when $\| u_0 \|_{H^1_{per}(\Omega)} < R$.

Proof. Multiply (1) by $-\Delta\bar{u}$, and then integrate it on Ω and taking the real part of the resulting identity, we find that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} \| \nabla u \|^2 = {} & \rho \| \nabla u \|^2 + \| \Delta u \|^2 \\ & - \nu \| \nabla\Delta u \|^2 + (\Delta\varphi(u), \Delta u) \\ & + Re(1 + i\mu)\int | u |^{2\sigma} u \Delta\bar{u} \\ & - Re(\Delta u, \alpha\lambda_1 \cdot \nabla(| u |^2 u) \\ & + \beta(\lambda_2 \cdot \nabla u) | u |^2). \end{aligned} \qquad (9)$$

First, we have

$$\begin{aligned}(\Delta\varphi(u), \Delta u) &= -(\nabla\varphi(u), \nabla\Delta u) \\ &= -(\varphi'(u)\nabla u, \nabla\Delta u) \\ &\leqslant c \| u \|_{4p}^p \| \nabla u \|_4 \| \nabla\Delta u \|.\end{aligned}$$

Noting that G-N' inequality

$$\| u \|_r \leqslant c \| \nabla\Delta u \|^{\frac{r-2}{2r}} \| u \|^{\frac{r+2}{2r}}, \qquad (10)$$

$$\| \nabla u \|_r \leqslant c \| \nabla\Delta u \|^{\frac{5r-6}{6r}} \| u \|^{\frac{r+6}{6r}}, \qquad (11)$$

we get

$$\begin{aligned}(\Delta\varphi(u), \Delta u) &\leqslant c \| \nabla\Delta u \|^{\frac{3p+8}{6}} \| u \|^{\frac{3p+4}{6}} \\ &\qquad \text{since } 0 \leqslant p < \frac{4}{3} \\ &\leqslant \frac{\nu}{6} \| \nabla\Delta u \|^2 + c \quad \text{by Lemma 5.}\end{aligned} \qquad (12)$$

Since

$$\begin{aligned}| u |^2 | \nabla u |^2 = {} & \frac{1}{4} | \nabla | u |^2 |^2 \\ & + \frac{1}{4} | u \nabla\bar{u} - \bar{u}\nabla u |^2,\end{aligned} \qquad (13)$$

hereafter, we will make use of the identity.

The fifth term in (9) on the right is changed to

$$\begin{aligned}&Re(1 + i\mu)\int | u |^{2\sigma} u \Delta\bar{u} \\ &= -Re(1 + i\mu)\int | u |^{2\sigma} | \nabla u |^2 \\ &\quad - Re(1 + i\mu)\sigma\int | u |^{2\sigma-2} u \nabla\bar{u} \nabla | u |^2 \\ &= -\int | u |^{2\sigma} | \nabla u |^2 \\ &\quad - \frac{\sigma}{2}\int | u |^{2\sigma} | \nabla | u |^2 |^2 \\ &\quad + \frac{1}{2}\mu\sigma\int | u |^{2\sigma} \nabla | u |^2 \\ &\quad \cdot i(u\nabla\bar{u} - \bar{u}\nabla u) \\ &= -\frac{1}{4}\int | u |^{2\sigma-2}((1 + 2\sigma) | \nabla | u |^2 |^2 \\ &\quad - 2\mu\sigma \nabla | u |^2 \cdot i(u\nabla\bar{u} - \bar{u}\nabla u) \\ &\quad + | u\nabla\bar{u} - \bar{u}\nabla u |^2).\end{aligned} \qquad (14)$$

The integrand in (14) is a quadratic form in these quantities that will be nonnegative provided the matrix

$$\begin{pmatrix} 1+2\sigma & \sigma\mu \\ \sigma\mu & 1 \end{pmatrix}$$

is nonnegative define, i.e.

$$\sigma < \frac{1}{\sqrt{1+\mu^2}-1}.$$

Noting

$$\| \nabla u \| \leqslant \| \Delta u \|^{\frac{1}{2}} \| u \|^{\frac{1}{2}},$$

we deduce that

$$\| \Delta u \| \leqslant \| \nabla \Delta u \|^{\frac{1}{2}} \| \nabla u \|^{\frac{1}{2}}. \tag{15}$$

Using G-N' inequality, we deduce

$$\| u \|_{L^4} \leqslant \| \Delta u \|^{\frac{3}{8}} \| u \|^{\frac{5}{8}}, \tag{16}$$

$$\| u \|_{L^8} \leqslant \| \nabla \Delta u \|^{\theta} \| u \|_q^{1-\theta}, \tag{17}$$

where $\theta = \frac{8-q}{4(2+q)}$, for $1 < q < 8$; $\theta = 0$ for $q \geqslant 8$. And we deduce

$$\| \nabla u \|_{L^4} \leqslant \| \nabla \Delta u \|^{\frac{3}{8}} \| \nabla u \|^{\frac{5}{8}}. \tag{18}$$

Hence, for the sixth term of (9) on the right side we deduce

$$\begin{aligned}
&| - Re(\Delta u, \beta(\lambda_2 \cdot \nabla u) | u |^2) | \\
&\leqslant | \beta\lambda_2 | \int | \Delta u | | \nabla u | | u |^2 \\
&\leqslant | \beta\lambda_2 | \, \| \Delta u \| \, \| \nabla u \|_4 \| u \|_8^2 \\
&\leqslant | \beta\lambda_2 | \, \| \nabla \Delta u \|^{\frac{1}{2}} \| \nabla u \|^{\frac{1}{2}} \| \nabla \Delta u \|^{\frac{3}{8}} \\
&\quad \cdot \| \nabla u \|^{\frac{5}{8}} (\| \nabla \Delta u \|^{\theta} \| u \|_q^{1-\theta})^2 \\
&\leqslant c \, | \beta\lambda_2 | \, \| \nabla \Delta u \|^{\frac{7}{8}+2\theta} \\
&\quad \cdot \| \nabla u \|^{\frac{9}{8}} \| u \|_q^{2(1-\theta)} \\
&\leqslant \frac{\nu}{6} \| \nabla \Delta u \|^2 + c \, | \beta\lambda_2 |^{\frac{16}{9-16\theta}} \\
&\quad \cdot \| \nabla u \|^{\frac{18}{9-16\theta}} \| u \|_q^{\frac{32(1-\theta)}{9-16\theta}} \\
&\quad \left(\text{as } \theta = \frac{8-q}{4(2+q)}, \right. \\
&\quad \left. \text{and } 9 - 16\theta > 0 \text{ then } q > \frac{14}{13} \right) \\
&\leqslant \frac{\nu}{6} \| \nabla \Delta u \|^2 + c \| \nabla u \|^4 \\
&\quad + c \, | \beta\lambda_2 |^{\frac{32}{9-32\theta}} \| u \|_q^{\frac{64(1-\theta)}{9-32\theta}} \\
&\quad \left(\text{as } \theta = \frac{8-q}{4(2+q)}, \right. \\
&\quad \left. \text{and } 9 - 32\theta > 0 \text{ then } q > \frac{46}{17} \right) \\
&\leqslant \frac{\nu}{6} \| \nabla \Delta u \|^2 + c \| \nabla u \|^4 \\
&\quad + c \| u \|_q^q + c \, | \beta\lambda_2 |^{\frac{32q}{9q-32q\theta-64+64\theta}} \\
&\quad \left(\text{as } \theta = \frac{8-q}{4(2+q)}, \text{ and} \right. \\
&\quad \left. 9q - 32q\theta - 64 + 64\theta > 0 \text{ then } q > \frac{126}{17} \right) \\
&\leqslant \frac{\nu}{6} \| \nabla \Delta u \|^2 + c \| \nabla u \|^4 \\
&\quad + c \| u \|_{2\sigma+2}^{2\sigma+2} + c \, | \beta\lambda_2 |^{\frac{64(\sigma+1)(\sigma+2)}{17\sigma-49}} \\
&\quad \left(\text{Let } q = 2\sigma + 2. \text{ By } \sigma \geqslant 3 > \frac{46}{17} \right. \\
&\quad \left. \text{we have } q = 2\sigma + 2 > \frac{126}{17} \right).
\end{aligned} \tag{19}$$

Similarly, due to

$$\begin{aligned}
\nabla(| u |^2 u) &= | u |^2 \nabla u + \nabla | u |^2 u \\
&= 2 | u |^2 \nabla u + u^2 \nabla \bar{u},
\end{aligned}$$

we infer that

$$\begin{aligned}
&| - Re(\Delta u, \alpha\lambda_1 \cdot \nabla(| u |^2 u)) | \\
&\leqslant \frac{\nu}{6} \| \nabla \Delta u \|^2 + c \| \nabla u \|^4 + c \| u \|_{2\sigma+2}^{2\sigma+2} \\
&\quad + c \, | \alpha\lambda_1 |^{\frac{64(\sigma+1)(\sigma+2)}{17\sigma-49}}.
\end{aligned} \tag{20}$$

We give from Eqs. (9), (12), (14), (19), (20) that

$$\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \| \nabla u \|^2 + \frac{\nu}{2} \| \nabla \Delta u \|^2 \\
&\leqslant \rho \| \nabla u \|^2 + \| \Delta u \|^2 \\
&\quad + c \| \nabla u \|^4 + c \| u \|_{2\sigma+2}^{2\sigma+2} + c \\
&\quad + c(| \beta\lambda_2 | + | \alpha\lambda_1 |)^{\frac{64(\sigma+1)(\sigma+2)}{17\sigma-49}}.
\end{aligned} \tag{21}$$

By Lemma 5, we have

$$\begin{aligned}
c \| \nabla u \|^2 &= c(-\Delta u, u) \\
&\leqslant c \| \Delta u \| \, \| u \| \leqslant k \| \Delta u \|,
\end{aligned}$$

where k depends only on the data, and thus we deduce that

$$\begin{aligned}
\| \Delta u \|^2 + c \| \nabla u \|^4 &\leqslant (1 + k^2) \| \Delta u \|^2 \\
&= -(1 + k^2)(\nabla \Delta u, \nabla u) \\
&\leqslant (1 + k^2) \| \nabla \Delta u \| \, \| \nabla u \| \\
&\leqslant \frac{\nu}{4} \| \nabla \Delta u \|^2 + \frac{2}{\nu}(1 + k^2)^2 \| \nabla u \|^2.
\end{aligned} \tag{22}$$

We obtain from Eqs. (21), (22) that

$$\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \| \nabla u \|^2 + \frac{\nu}{4} \| \nabla \Delta u \|^2 \\
&\leqslant \left(\rho + \frac{2}{\nu}(1 + k^2)^2 \right) \| \nabla u \|^2 + c \| u \|_{2\sigma+2}^{2\sigma+2} \\
&\quad + c + c(| \beta\lambda_2 | + | \alpha\lambda_1 |)^{\frac{64(\sigma+1)(\sigma+2)}{17\sigma-49}}.
\end{aligned} \tag{23}$$

By (5) and

$$\|\nabla u\|^2 = -(\Delta u, u) \leqslant \|\Delta u\|^2 + c\|u\|^2,$$

we deduce that

$$\int_t^{t+1}\|\nabla u\|^2 dt \leqslant \int_t^{t+1}\|\Delta u\|^2 dt + c \leqslant K_1 + c, \quad t \geqslant t_1.$$

Noting (6), and applying uniform Gronwall's inequality to (23), we obtain the proof of Lemma 6.

Lemma 7. For solution $u(t)$ of (1)~(3) we have

$$\frac{1}{2(1+\sigma)}\frac{d}{dt}\int |u|^{2\sigma+2} + \frac{1}{2}\|u\|_{4\sigma+2}^{4\sigma+2} + \nu\int \Delta^2 u\,|u|^{2\sigma}\bar{u} \leqslant \varepsilon\|\Delta^2 u\|^2 + \varepsilon\|\nabla u\|^4 + c + c(\varepsilon)(|\alpha\lambda_1| + |\beta\lambda_2|)^{\frac{16(2\sigma+1)(5\sigma+4)}{27\sigma-52}}. \quad (24)$$

Proof. Multiplying (1) by $|u|^{2\sigma}\bar{u}$ then integrating it on Ω and taking the real part of the resulting identity, we find that

$$\begin{aligned}\frac{1}{2(1+\sigma)}\frac{d}{dt}\int |u|^{2\sigma+2} &= \rho\int |u|^{2\sigma+2} + Re(1+i\gamma)\int \Delta u\,|u|^{2\sigma}\bar{u} \\ &+ (\Delta\varphi(u), |u|^{2\sigma}u) \\ &- \nu\int \Delta^2 u\,|u|^{2\sigma}\bar{u} - \int |u|^{4\sigma+2} \\ &+ Re(|u|^{2\sigma}u, \alpha\lambda_1\cdot\nabla(|u|^2 u) \\ &+ \beta(\lambda_2\cdot\nabla u)|u|^2). \end{aligned} \quad (25)$$

First

$$\begin{aligned}(\Delta\varphi(u), |u|^{2\sigma}\bar{u}) &\leqslant \|\Delta\varphi(u)\|\,\|u\|_{4\sigma+2}^{2\sigma+1} \\ &\leqslant \|\varphi''(u)(\nabla u)^2 + \varphi'(u)\Delta u\|\,\|u\|_{4\sigma+2}^{2\sigma+1} \\ &\leqslant (c\|u\|_{4(p-1)}^{p-1}\|\nabla u\|_8^2 \\ &+ c\|u\|_{4p}^{p}\|\Delta u\|_4)\|u\|_{4\sigma+2}^{2\sigma+1}.\end{aligned}$$

Because of G-N's inequality

$$\|\Delta u\|_4 \leqslant c\|\Delta^2 u\|^{\frac{7}{12}}\|u\|^{\frac{5}{12}},$$

$$\|\nabla u\|_8 \leqslant c\|\Delta^2 u\|^{\frac{3}{8}}\|u\|^{\frac{5}{8}},$$

we have

$$\begin{aligned}(\Delta\varphi(u), |u|^{2\sigma}\bar{u}) &\leqslant (c\|u\|_{4(p-1)}^{p-1}\|\nabla u\|_8^2 \\ &+ c\|u\|_{4p}^{p}\|\Delta u\|_4)\|u\|_{4\sigma+2}^{2\sigma+1} \\ &\leqslant \frac{1}{8}\|u\|_{4\sigma+2}^{4\sigma+2} \\ &+ c\|\Delta^2 u\|^{\frac{3+6p}{8}}\|\nabla u\|^{\frac{5}{2}}\|u\|^{\frac{10p-7}{8}} \\ &+ c\|\Delta^2 u\|^{\frac{19+18p}{24}}\|\nabla u\|^{\frac{5}{6}}\|u\|^{\frac{10p+3}{8}} \end{aligned}$$

since $p < \frac{4}{3}$

$$\leqslant \frac{1}{8}\|u\|_{4\sigma+2}^{4\sigma+2} + \varepsilon\|\Delta^2 u\|^2 + c, \quad (26)$$

where we have used Young's inequality, and Lemmas 5 and 6.

Owing to (13), the second term in (25) on the right side is changed to

$$\begin{aligned}Re(1+i\gamma)\int \Delta u\,|u|^{2\sigma}\bar{u} &= -Re(1+i\gamma)\int |\nabla u|^2|u|^{2\sigma} \\ &- Re(1+i\gamma)\int \sigma|u|^{2\sigma-2}\bar{u}\nabla u.\nabla|u|^2 \\ &= -\int |\nabla u|^2|u|^{2\sigma} \\ &- \frac{\sigma}{2}\int |u|^{2\sigma-2}|\nabla|u|^2|^2 \\ &+ \frac{1}{2}\gamma\sigma\int |u|^{2\sigma-2}\nabla|u|^2 \\ &\cdot i(u\nabla\bar{u} - \bar{u}\nabla u) \\ &= -\frac{1}{4}\int |u|^{2\sigma-2}((1+2\sigma)|\nabla|u|^2|^2 \\ &- 2\gamma\sigma\nabla|u|^2\cdot i(u\nabla\bar{u} - \bar{u}\nabla u) \\ &+ |u\nabla\bar{u} - \bar{u}\nabla u|^2). \end{aligned} \quad (27)$$

By Young's inequality and Lemma 5, the first term in (25) on the right is changed to

$$\begin{aligned}\rho\int |u|^{2\sigma+2} &= \rho\int |u|^{2\sigma+1}\cdot|u| \\ &\leqslant \frac{1}{8}\int |u|^{4\sigma+2} + 2\rho^2\int |u|^2 \\ &\leqslant \frac{1}{8}\int |u|^{4\sigma+2} + c, \quad \forall t > t_1. \end{aligned} \quad (28)$$

By G-N's inequality, we deduce

$$\|u\|_{L^4} \leqslant \|\nabla\Delta u\|^{\frac{1}{4}}\|u\|^{\frac{3}{4}},$$

$$\|u\|_{L^8} \leqslant \|\Delta^2 u\|^{\theta}\|u\|_q^{1-\theta},$$

where $\theta = \frac{24-3q}{4(6+5q)}$, for $1<q<8$; $\theta=0$ for $q\geqslant 8$. And we deduce

$$\|\nabla u\|_{L^4} \leqslant \|\Delta^2 u\|^{\frac{1}{4}}\|\nabla u\|^{\frac{3}{4}}.$$

Thus, the fourth term in (25) on the right is changed to

$$\begin{aligned}|Re(|u|^{2\sigma}u, \beta(\lambda_2\cdot\nabla u)|u|^2)| &= \left|Re\beta\int(\lambda_2\cdot\nabla u)|u|^{2\sigma+2}\bar{u}\right| \\ &\leqslant |\beta\lambda_2|\int |\nabla u||u|^2|u|^{2\sigma+1} \\ &\leqslant 3|\beta\lambda_2|^2\int |\nabla u|^2|u|^4 + \frac{1}{16}\int |u|^{4\sigma+2}\end{aligned}$$

$$\leqslant 3 |\beta\lambda_2|^2 \|\nabla u\|_4^2 \|u\|_8^4 + \frac{1}{16}\int |u|^{4\sigma+2}$$
$$\leqslant 3 |\beta\lambda_2|^2 (\|\Delta^2 u\|^{\frac{1}{4}} \|\nabla u\|^{\frac{3}{4}})^2 \cdot (\|\Delta^2 u\|^{\theta} \|u\|_q^{1-\theta})^4 + \frac{1}{16}\int |u|^{4\sigma+2}$$
$$= 3 |\beta\lambda_2|^2 \|\Delta^2 u\|^{\frac{1}{2}+4\theta} \|\nabla u\|^{\frac{3}{2}} \cdot \|u\|_q^{4(1-\theta)} + \frac{1}{16}\int |u|^{4\sigma+2}$$
$$\leqslant \varepsilon \|\Delta^2 u\|^2 + c(\varepsilon) |\beta\lambda_2|^{\frac{4}{3-8\theta}} \cdot \|\nabla u\|^{\frac{6}{3-8\theta}} \|u\|_q^{\frac{16(1-\theta)}{3-8\theta}} + \frac{1}{16}\int |u|^{4\sigma+2}$$
$$\left(\text{as } \theta = \frac{24-3q}{4(6+5q)}, \text{ and } 3-8\theta > 0 \text{ then } q > \frac{10}{7}\right)$$
$$\leqslant \varepsilon \|\Delta^2 u\|^2 + \varepsilon \|\nabla u\|^4 + c(\varepsilon) |\beta\lambda_2|^{\frac{8}{3-16\theta}} \|u\|_q^{\frac{32(1-\theta)}{3-16\theta}} + \frac{1}{16}\int |u|^{4\sigma+2}$$
$$\left(\text{as } \theta = \frac{24-3q}{4(6+5q)}, \text{ and } 3-16\theta > 0 \text{ then } q > \frac{26}{9}\right)$$
$$\leqslant \varepsilon \|\Delta^2 u\|^2 + \varepsilon \|\nabla u\|^4 + \frac{1}{16}\|u\|_q^q + c(\varepsilon) |\beta\lambda_2|^{\frac{8q}{3q-16q\theta-32+32\theta}} + \frac{1}{16}\int |u|^{4\sigma+2}$$
$$\left(\text{as } \theta = \frac{24-3q}{4(6+5q)}, \text{ and } 3q-16q\theta-32+32\theta > 0 \text{ then } q > \frac{262}{27}\right)$$
$$\leqslant \varepsilon \|\Delta^2 u\|^2 + \varepsilon \|\nabla u\|^4 + \frac{1}{16}\|u\|_{4\sigma+2}^{4\sigma+2} + c(\varepsilon) |\beta\lambda_2|^{\frac{16(2\sigma+1)(5\sigma+4)}{27\sigma-52}} + \frac{1}{16}\int |u|^{4\sigma+2}$$
$$\left(\text{Let } q = 4\sigma+2. \text{ By } \sigma \geqslant 3 > \frac{52}{27} \text{ we have } q = 4\sigma+2 > \frac{262}{27}\right)$$
$$\leqslant \varepsilon \|\Delta^2 u\|^2 + \varepsilon \|\nabla u\|^4 + \frac{1}{8}\|u\|_{4\sigma+2}^{4\sigma+2} + c(\varepsilon) |\beta\lambda_2|^{\frac{16(2\sigma+1)(5\sigma+4)}{27\sigma-52}}. \quad (29)$$

Due to

$$\nabla(|u|^2 u) = |u|^2 \nabla u + \nabla |u|^2 u = 2|u|^2 \nabla u + u^2 \nabla \bar{u},$$

and similar to the above estimate, we infer that

$$|Re(|u|^{2\sigma} u, \alpha\lambda_1 \cdot \nabla(|u|^2 u))| \leqslant 3 |\alpha\lambda_1| \int |u|^{2\sigma+1} |u|^2 |\nabla u|$$
$$\leqslant \varepsilon \|\Delta^2 u\|^2 + \varepsilon \|\nabla u\|^4 + \frac{1}{8}\|u\|_{4\sigma+2}^{4\sigma+2} + c(\varepsilon) |\alpha\lambda_1|^{\frac{16(2\sigma+1)(5\sigma+4)}{27\sigma-52}}. \quad (30)$$

By Eqs. (25)~(30), we obtain

$$\frac{1}{2(1+\sigma)}\frac{d}{dt}\int |u|^{2\sigma+2} + \frac{1}{2}\|u\|_{4\sigma+2}^{4\sigma+2} + \nu\int \Delta^2 u |u|^{2\sigma}\bar{u}$$
$$\leqslant \varepsilon \|\Delta^2 u\|^2 + \varepsilon \|\nabla u\|^4 + c + c(\varepsilon)(|\alpha\lambda_1| + |\beta\lambda_2|)^{\frac{16(2\sigma+1)(5\sigma+4)}{27\sigma-52}}$$
$$- \frac{1}{4}\int |u|^{2\sigma-2}((1+2\sigma)|\nabla|u|^2|^2 - 2\gamma\sigma \nabla|u|^2 \cdot i(u\nabla\bar{u} - \bar{u}\nabla u) + |u\nabla\bar{u} - \bar{u}\nabla u|^2) + c. \quad (31)$$

The integrand in the last term in (17) is a quadratic form in these quantities that will be nonnegative provided the matrix

$$\begin{pmatrix} 1+2\sigma & \sigma\gamma \\ \sigma\gamma & 1 \end{pmatrix}$$

is nonnegative define, i.e.

$$\sigma < \frac{1}{\sqrt{1+\gamma^2}-1}.$$

Thus, we complete the proof of Lemma 7.

Lemma 8. Assume that the conditions of Lemma 6 hold, and

$$|\varphi''(u)| \leqslant c |u|^{p-1},$$

where p is the same as in Lemma 6. Then for $u_0 \in H_{per}^2(\Omega)$, the solution of problems (1)~(3) satisfies

$$\|\Delta u\|^2 \leqslant K_3(T) \text{ for } t \in [0, T],$$

where K_3 depends on T, and T depends on R when $\|u_0\|_{H_{per}^2(\Omega)} \leqslant R$, $\|u_0\|_{L^{2\sigma+2}} \leqslant R$.

Proof. Taking the inner product of (1) with $\Delta^2 u$, and taking real part of the resulting identity, we find that

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 = \rho\|\Delta u\|^2 + \|\nabla\Delta u\|^2 - \nu\|\Delta^2 u\|^2 - (\Delta\varphi(u), \Delta^2 u) - Re(1+i\mu)\int |u|^{2\sigma} u \Delta^2\bar{u} + (\Delta^2 u, \alpha\lambda_1 \cdot \nabla(|u|^2 u))$$

$$+\beta(\lambda_2\cdot\nabla u)|u|^2). \tag{32}$$

We estimate each term of (32) on the right side now.

We get

$$\begin{aligned}-(\Delta\varphi(u),\Delta^2 u) &\leqslant \|\Delta\varphi(u)\|\ \|\Delta^2 u\|\\ &\leqslant \|\varphi''(u)(\nabla u)^2+\varphi'(u)\Delta u\|\ \|\Delta^2 u\|\\ &\leqslant c\|u\|_{4(p-1)}^{p-1}\|\nabla u\|_8^2\|\Delta^2 u\|\\ &\quad + c\|u\|_{4p}^{p}\|\Delta u\|_4\|\Delta^2 u\|.\end{aligned}$$

Because of G-N' inequality

$$\|\Delta u\|_4\leqslant c\|\Delta^2 u\|^{\frac{7}{12}}\|u\|^{\frac{5}{12}},$$

$$\|\nabla u\|_8\leqslant c\|\Delta^2 u\|^{\frac{3}{8}}\|u\|^{\frac{5}{8}},$$

we have

$$\begin{aligned}-(\Delta\varphi(u),\Delta^2 u) &\leqslant c\|u\|_{4(p-1)}^{p-1}\|\nabla u\|_8^2\|\Delta^2 u\|\\ &\quad + c\|u\|_{4p}^{p}\|\Delta u\|_4\|\Delta^2 u\|\\ &\leqslant c\|\Delta^2 u\|^{1+\frac{3+6p}{16}}\|\nabla u\|^{\frac{5}{4}}\|u\|^{\frac{10p-7}{16}}\\ &\quad + c\|\Delta^2 u\|^{1+\frac{19+18p}{48}}\|\nabla u\|^{\frac{5}{12}}\|u\|^{\frac{10p+3}{16}}\\ &\leqslant c\|\Delta^2 u\|^{1+\frac{3+6p}{16}}+c\|\Delta^2 u\|^{1+\frac{19+18p}{48}}\\ &\leqslant \frac{\nu}{6}\|\Delta^2 u\|^2+c,\end{aligned} \tag{33}$$

where Young's inequality, Lemmas 4 and 6 are used.

By G-N's inequality

$$\|u\|_{L^4}\leqslant \|\nabla\Delta u\|^{\frac{1}{4}}\|u\|^{\frac{3}{4}},$$

$$\|u\|_{L^8}\leqslant \|\Delta^2 u\|^{\frac{9}{32}}\|u\|^{\frac{23}{32}},$$

for the fifth term of (32) we estimate

$$\begin{aligned}(\Delta^2 u,\alpha\lambda_1\cdot\nabla(|u|^2 u)) &\leqslant |\alpha\lambda_1|\left|\int\Delta^2 u\,\nabla(|u|^2 u)\right|\\ &\leqslant 3|\alpha\lambda_1|\int|\Delta^2 u||u|^2|\nabla u|\\ &\leqslant 3|\alpha\lambda_1|\ \|\Delta^2 u\|\ \|u\|_8^2\|\nabla u\|_4\\ &\leqslant 3|\alpha\lambda_1|\ \|\Delta^2 u\|^{1+\frac{13}{16}}\|\nabla u\|^{\frac{3}{4}}\|u\|^{\frac{23}{16}}\\ &\leqslant \frac{\nu}{6}\|\Delta^2 u\|^2+c,\end{aligned} \tag{34}$$

where Lemmas 5 and 6 are used.

Similar to (34), we have

$$|(\Delta^2 u,\beta(\lambda_2\cdot\nabla u)|u|^2)|\leqslant\frac{\nu}{6}\|\Delta^2 u\|^2+c. \tag{35}$$

Therefore, combining (32) with (35), we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|^2+\frac{\nu}{2}\|\Delta^2 u\|^2 &\leqslant \rho\|\Delta u\|^2+\|\nabla\Delta u\|^2+c\\ &\quad - Re(1+i\mu)\int|u|^{2\sigma}u\Delta^2\bar{u}.\end{aligned} \tag{36}$$

Taking a linear combination of (24) and (36), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Delta u\|^2+\frac{\delta^2}{1+\sigma}\int|u|^{2\sigma+2}\right)\\ &\quad +\frac{\nu}{2}\|\Delta^2 u\|^2+\frac{\delta^2}{2}\|u\|_{4\sigma+2}^{4\sigma+2}\\ &\leqslant \delta^2\varepsilon\|\Delta^2 u\|^2+\rho\|\Delta u\|^2\\ &\quad +\|\nabla\Delta u\|^2+\varepsilon\delta^2\|\nabla u\|^4\\ &\quad - Re(\nu\delta^2+1+i\mu)\int\Delta^2 u|u|^{2\sigma}\bar{u}+c\\ &\quad + c(\varepsilon)(|\alpha\lambda_1|+|\beta\lambda_2|)^{\frac{16(2\sigma+1)(5\sigma+4)}{27\sigma-52}}.\end{aligned} \tag{37}$$

When ε is small enough, and noting G-N's inequality, we easily obtain

$$\begin{aligned}&\delta^2\varepsilon\|\Delta^2 u\|^2+\rho\|\Delta u\|^2+\|\nabla\Delta u\|^2\\ &\quad +\varepsilon\delta^2\|\nabla u\|^4\leqslant\frac{\nu}{4}\|\Delta^2 u\|^2+c.\end{aligned} \tag{38}$$

For the fifth term of (37) on the right side, we obtain

$$\begin{aligned}&Re(\nu\delta^2+1+i\mu)\int|u|^{2\sigma}u\Delta^2\bar{u}\\ &\leqslant\sqrt{(1+\nu\delta^2)^2+\mu^2}\,\|u\|_{4\sigma+2}^{2\sigma+1}\|\Delta^2 u\|\\ &\leqslant\frac{\nu}{4}\|\Delta^2 u\|^2\\ &\quad +\frac{1}{\nu}((1+\nu\delta^2)^2+\mu^2)\|u\|_{4\sigma+2}^{4\sigma+2}.\end{aligned} \tag{39}$$

Let

$$k=\frac{\delta^2}{2}-\frac{1}{\nu}((1+\nu\delta^2)^2+\mu^2).$$

Noting condition (**A**)(ii), we obtain from (37) ~ (39) that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Delta u\|^2+\frac{\delta^2}{1+\sigma}\int|u|^{2\sigma+2}\right)\\ &\quad +\frac{\nu}{4}\|\Delta^2 u\|^2+k\|u\|_{4\sigma+2}^{4\sigma+2}\\ &\leqslant c+c(|\alpha\lambda_1|+|\beta\lambda_2|)^{\frac{16(2\sigma+1)(5\sigma+4)}{27\sigma-52}}.\end{aligned} \tag{40}$$

We also obtain very easily

$$\frac{\nu}{4}\|\Delta u\|^2\leqslant\|\Delta^2 u\|^2+c,$$

$$k \| u \|_{2\sigma+2}^{2\sigma+2} = k \int |u|^{2\sigma+1} |u| \leqslant \frac{\delta^2}{1+\sigma} \| u \|_{4\sigma+2}^{4\sigma+2} + c.$$

Therefore, we infer

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\| \Delta u \|^2 + \frac{\delta^2}{1+\sigma} \int |u|^{2\sigma+2} \right) + \left(\| \Delta u \|^2 + \frac{\delta^2}{1+\sigma} \int |u|^{2\sigma+2} \right) \leqslant c + c(|\alpha \lambda_1| + |\beta \lambda_2|)^{\frac{16(2\sigma+1)(5\sigma+4)}{27\sigma-52}} \quad \forall t \geqslant t^*, \tag{41}$$

where $t^* = \mathrm{Max}\{t_1, t_2\}$, and t_1, t_2 are the same as in Lemmas 5 and 6.

It follows from Gronwall's inequality that

$$\| \Delta u \|^2 + \frac{\delta^2}{1+\sigma} \int |u|^{2\sigma+2} \leqslant K_3(T), \quad \forall t \in [0, T],$$

which completes the proof of Lemma 8.

From Lemmas 5 ~ 8, we obtain the existence of global solution of the problems (1) ~ (3).

Theorem 9. (Global existence) Assume that the conditions of Lemmas 5 ~ 8 hold. Then for any $T > 0$, there exists a unique global solution of initial-value problem for the 3-D generalized complex Ginzburg-Landau equations (1) ~ (3), such that

$$u \in C^1([0, T]; H_{\mathrm{per}}^2(\Omega)) \cap C([0, T]; H_{\mathrm{per}}^2(\Omega)) \cap L^\infty([0, T]; L_{\mathrm{per}}^{2\sigma+2}(\Omega)).$$

References

1 Doering, C. et al. Low-dimensional behavior in the complex Ginzburg-Landau equation. Nonlinearity, 1988, 1: 279.

2 Ghidaglia, J. M. et al. Dimension of the attractor associated to the Ginzburg-Landau equation. Phys. D, 1987, 28: 282.

3 Bartuccelli, M. et al. On the possibility of soft and hard turbulence in the complex Ginzburg-Landau equation. Phys. D, 1990, 44: 421.

4 Doering, C. R. et al. Weak and strong solutions of the complex Ginzburg-Landau equation. Phys. D, 1994, 71: 285.

5 Guo, B. et al. Finite dimensional behavior of the generalized Ginzburg-Landau equation. Progress in Natural Science, 1994, 4: 423.

6 Henry, D. Geometric Theory of Semilinear Parabolic Equation. Berlin: Springer-Verlag, 1981.

7 Pazy, A. Semigroups of Linear Operators and Applications to Partial Differential Equation. Berlin: Springer-Verlag, 1983.